week3

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1 The Foundation of Applied Machine Learning

1.1 Spring 2019

1.1.1 Instructor: Prof. Bahram Mobasher

1.1.2 Teaching Assistance: Abtin Shahidi email *abtin.shahidi–at–email.ucr.edu*

1.1.3 Course webpage: https://abtinshahidi.github.io/teaching/2019-spring-foundation-machine-learning

2 Week 3

2.1 Continuing on Statistics and python:

2.1.1 In the following exercise we are going to make some assumption about data, make a model, and fit the parameters of the model:

3 Flip A Coin: Is this Coin Really Fair?

3.1 Imagine that you want to measure the fairness of a given coin:

3.1.1 You run the following experiment:

1. You count number of heads per $N = 20$ coin toss.
2. You do the previous line 100 times.

And this is the outcome of the experiment:

```
In [1]: data_array = [ 6,  7,  8, 11,  8,  7,  8,  9,  8,  5, 12,  7,  5,  8,  8,  8, 10,
                       9,  9,  7,  5, 11,  6,  2,  9,  8, 11,  8, 10,  5,  9, 11,  8,  9,
                       7,  8,  6,  8, 12,  9, 11,  9,  6,  7, 11,  5,  9,  6,  8, 12,  6,
                       8,  7,  8,  8, 11,  5,  6,  6,  7, 12,  9,  7,  8,  9,  7, 11,  7,
                       9,  4,  8,  9,  9,  9, 12,  6,  8,  7, 10,  6,  5,  8,  9,  7,  8,
                       7,  9,  7,  7, 12,  9, 11,  6,  5,  9,  7,  9,  7, 11,  8]
```

First we need to come up with a model for the data. We need to find the probability of each outcome first, before getting into the estimation for fairness. Let's ask a simple questions: **What are the possible outcomes of a coin toss?**

The answer is: (Head, tail) or (0,1) or (True, False) or (win, lose)

So, if we assume that the probability of getting 1 is p and p is not going to change throughout the experiment. Also, by definition the probability of getting 0 is $q = 1 - p$. (p is a quantity we are looking for, since it is a measure for the fairness of the coin)

Let's say that we are going to toss the coins N times and we get n desired outcome. (e.g. **Head** is the desired outcome)

But, what are the chances of getting n out of N coin tosses?

n desired outcome probability is p^n; also we have $N - n$ undesired outcome during the experiment which means that the total probability of getting n 1s and $N - n$ 0s is $p^n q^{N-n}$

Also, we do not care about the **order** of the coin toss. (e.g. (1,0,0,0,1), (0,1,1,0,0), (1,1, 0, 0, 0) all are considered same outcome) So, we need to multiply the previous probability by number of configurations. (Number of ways you can choose n ones, and $N - n$ zeros; which is: $\binom{N}{n}$)

So the probability of the n heads out of N coin toss, when the probability of single **head** is p, is the following:

$$p(n|N,p) = \binom{N}{n} p^n (1-p)^{N-n} \tag{1}$$

Which is called the **binomial distribution**.

There is a pre-defined binomial function in scipy package. However, since we are still trying to get familiar with python, let's write the function ourself as below:

```
In [2]: # importing packages
        import numpy as np
        import matplotlib.pyplot as plt

        # use LaTeX fonts in the plot
        plt.rc('text', usetex=True)
        plt.rc('font', family='serif')
```

```
In [3]: def binomial(n=0, N=1, p=1):
            """
            This is the probability mass function for the binomial distribution.

            INPUT:
                    n: Number of desired outcome
                    N: Number of trials
                    p: Probability of a desired outcome for each separate coin toss

            OUTPUT:
                    Probability of getting n desired outcome, out of N trials,
                    when the probality of desired outcome is p
            """
            from math import factorial
            factor = factorial(N)/(factorial(n)*factorial(N-n))
            return factor*(p**n)*(1-p)**(N-n)
```

3.1.2 Now we need to make some assumption about prior distribution of p which is the quantity of the interest.

Since we have no other information about the coin before-hand we can assume a **Uniform** prior for p. So, let's sample from 10^5 values for p from this uniform distribution.

```
In [4]: number_of_points = 10**5
        prior_p = np.linspace(0,1, number_of_points)
```

Importing time() for getting a benchmark for different methods:

```
In [5]: from time import time
```

3.1.3 A simple for loop:

In the following cell, we are going to calculate the probability of getting all the values in the data-set, while using different p. Since, we can assume that the experiments are independent, we can simply multiply all the probabilities. Then looking for the p value which maximize that probability; or in other words, is the most likely value for p given our data-set.

You should notice that we are using the **Bayes' law** again; we are looking for $P(p|X)$ in which X is the whole data-set. But, we can turn that around and look for much simpler quantity, using Bayes' law: $P(p|X) \sim P(X|p)$

```
In [6]: N = 20

        prob_p_cat=np.zeros(number_of_points)

        ti=time()
        for i,p in enumerate(prior_p):
            prob=1
            for data in data_array:
                prob *= binomial(data, N, p)
            prob_p_cat[i] = prob
        tf=time()-ti

        print("For loop method for {} data points and {} simulations (sampling p) takes: {:10.3
```

```
For loop method for 100 data points and 100000 simulations (sampling p) takes:     24.183 secon
```

This is the most likely value according to the description above.

```
In [7]: prior_p[prob_p_cat==max(prob_p_cat)]
```

```
Out[7]: array([0.40300403])
```

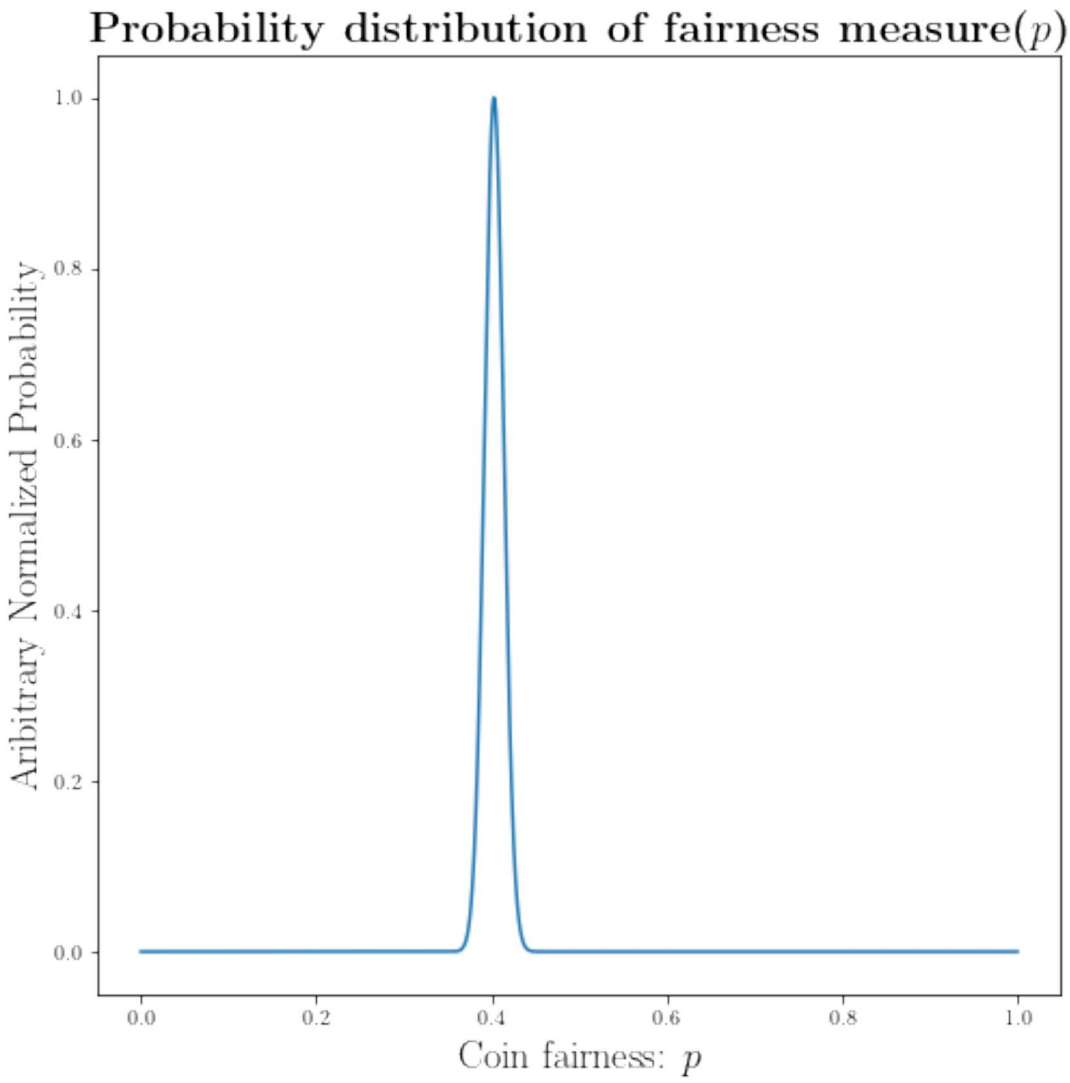
3.1.4 Using numpy.vectorize:

In this method instead of using a for loop on the elements of the data set we can use the numpy.vectorize(binomial), which allows us to give the vectorized function the whole array of data.

```
In [8]: vec_binomial = np.vectorize(binomial)

In [9]: N = 20

        prob_p_cat=np.zeros(number_of_points)

        ti=time()
        for i,p in enumerate(prior_p):
            prob_p_cat[i] = np.prod(vec_binomial(data_array, N, p))
        tf=time()-ti


        print("numpy.vectorize method for {} data points and {} simulations (sampling p) takes

numpy.vectorize method for 100 data points and 100000 simulations (sampling p) takes:     19.48


In [10]: prior_p[prob_p_cat==max(prob_p_cat)]

Out[10]: array([0.40300403])
```

You can see that the numpy.vectorize method is a little bit **faster** than the simple for loop.

```
In [11]: fig_p = plt.figure(figsize=(8,8))

         # Just normalizing the probability to maximum value so most likely
         # value corresponds to 1.
         # For getting the through probability we need to find the Integral
         # of the un-normalized distribution.

         plt.plot(prior_p, prob_p_cat/max(prob_p_cat),  markersize=1)

         plt.title(r"\textbf{Probability distribution of fairness measure($p$)}", fontsize=20)

         plt.xlabel(r"Coin fairness: $p$", fontsize=18)
         plt.ylabel(r"Aribitrary Normalized Probability", fontsize=18)


         plt.show()
```



As you can see from our experiment we found the distribution of the interest: $P(p|X)$

3.1.5 *Result:*

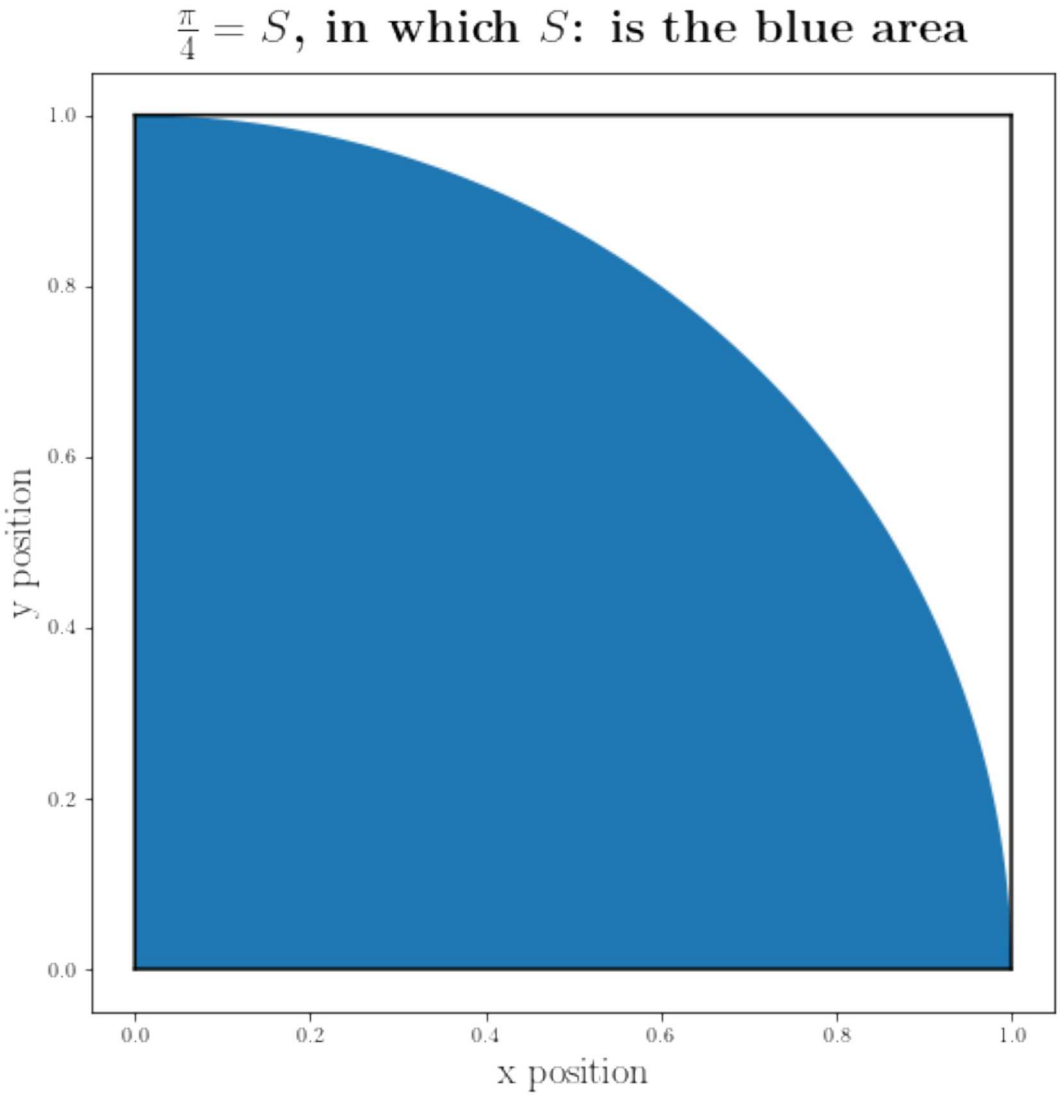
The **most likely value** for the fairness of the coin is 0.403, which shows that our coin is **biased** toward getting **tail**.

4 What is the numerical value of π?

4.1 There is a simple geometric approach we can use:

```
In [12]: fig_pi = plt.figure(figsize=(8,8))
```

```
from matplotlib.patches import Rectangle
_number_of_points_=10**6

x = np.linspace(0,1, _number_of_points_)
y = np.sqrt(1-x**2)
# Plot the circle
plt.fill_between(x,y)

# There are more sophisticated ways to do this as well!
# Making the square
y1 = np.ones(10**6)
y2 = np.zeros(10**6)

# make the square plot
plt.plot(x, y1, "k")
plt.plot(x, y2, "k")
plt.plot(y2, x, "k")
plt.plot(y1, x, "k")


plt.title(r"\textbf{$\frac{\pi}{4}=S$, in which $S$: is the blue area}", fontsize=22)

plt.xlabel(r"x position", fontsize=18)
plt.ylabel(r"y position", fontsize=18)


plt.show()
```



So if we somehow manage to find the blue area we can find the numerical value for π.

$$S = \frac{\pi R^2}{4} = \frac{\pi}{4} \quad \text{Since we know } R = 1$$
$$\pi = 4S \quad \text{So we need S}$$

4.1.1 How can we find the area numerically?

First we need to get familiar with the **Monte Carlo Simulations**

The **Monte Carlo** simulations, is a statistical technique to model stochastic (or probabilistic) systems and to find the probability of different outcome.

Further references: `http://mathworld.wolfram.com/MonteCarloMethod.html`

4.1.2 First let's start with making N random (x, y) points from $[0, 1]$ range:

```
In [13]: N = 10e4
         N = int(N)

         x = np.random.random(N)
         y = np.random.random(N)
```

Now we calculate the distance of each point from $(0, 0)$: distance function –> d((x,y), (0,0))

```
In [14]: distance_from_0_0 = np.sqrt(x**2 + y**2)
```

Let's count number of points with $d((x, y), (0, 0)) \leq 1$

```
In [15]: circle_points = distance_from_0_0[distance_from_0_0<=1]
```

Now we have an array of distances for points inside the circle ($d((x, y), (0, 0)) \leq 1$):
If we define n to be the number of points within circle, and N to be total number of points, We can find the area to be:

$$S = \frac{n}{N}$$

```
In [16]: PI = 4 * len(circle_points)/len(distance_from_0_0)

         print(PI)
```

```
3.14204
```

As you can see we are getting close to the **True value**.
Let's put the above procedures inside a function:

```
In [17]: def our_PI_generator(N=10e5):
             """This is our generic code for approximating pi ~ 3.14 with Monte Carlo simulati
             import numpy as np
             # initializing
             N = int(N)
             # Produce random numbers between [0,1] for (x,y)
             x = np.random.random(N)
             y = np.random.random(N)
             # Find the distance of (x,y) from [0,0]
             distance_from_0_0 = np.sqrt(x**2 + y**2)
             # imposing the condition for the circle:  distance((x,y),(0,0))<= 0
             circle_points = distance_from_0_0[distance_from_0_0<=1]
             return 4 * len(circle_points)/N



In [18]: our_PI_generator(10e7)
```

```
Out[18]: 3.1411142
```

Let's use different number of points to see how adding to the number of points changes our numerical estimate for π:

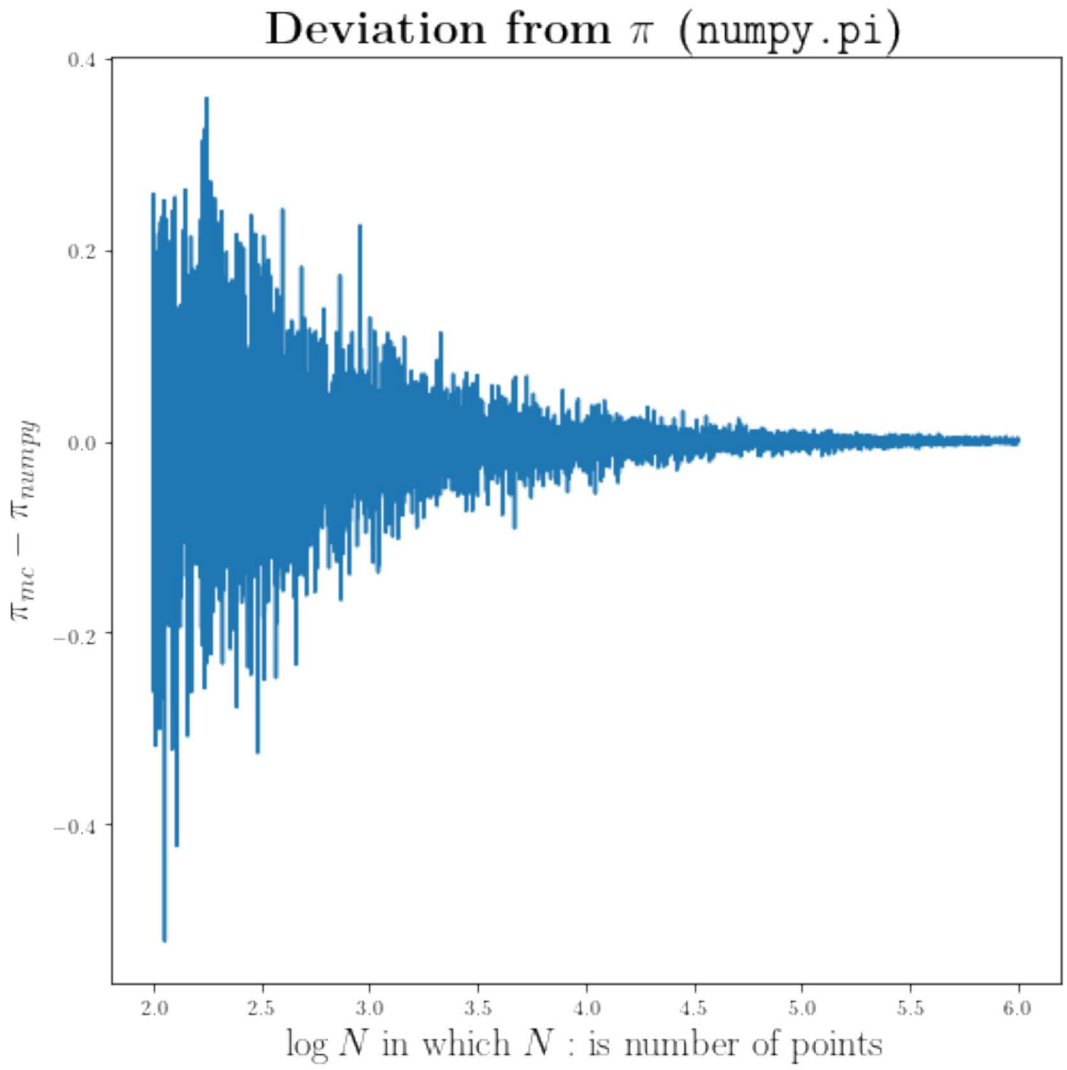
```
In [19]: np.logspace(2, 6, 10)

Out[19]: array([1.00000000e+02, 2.78255940e+02, 7.74263683e+02, 2.15443469e+03,
                5.99484250e+03, 1.66810054e+04, 4.64158883e+04, 1.29154967e+05,
                3.59381366e+05, 1.00000000e+06])

In [20]: I =  np.logspace(2, 6, 5000)
         I = np.array([int(i) for i in I])

         _x_ = [our_PI_generator(i) for i in I]
```

Here we assume the true value of π is coming from `numpy.pi`. Let's find the errors of our estimates:

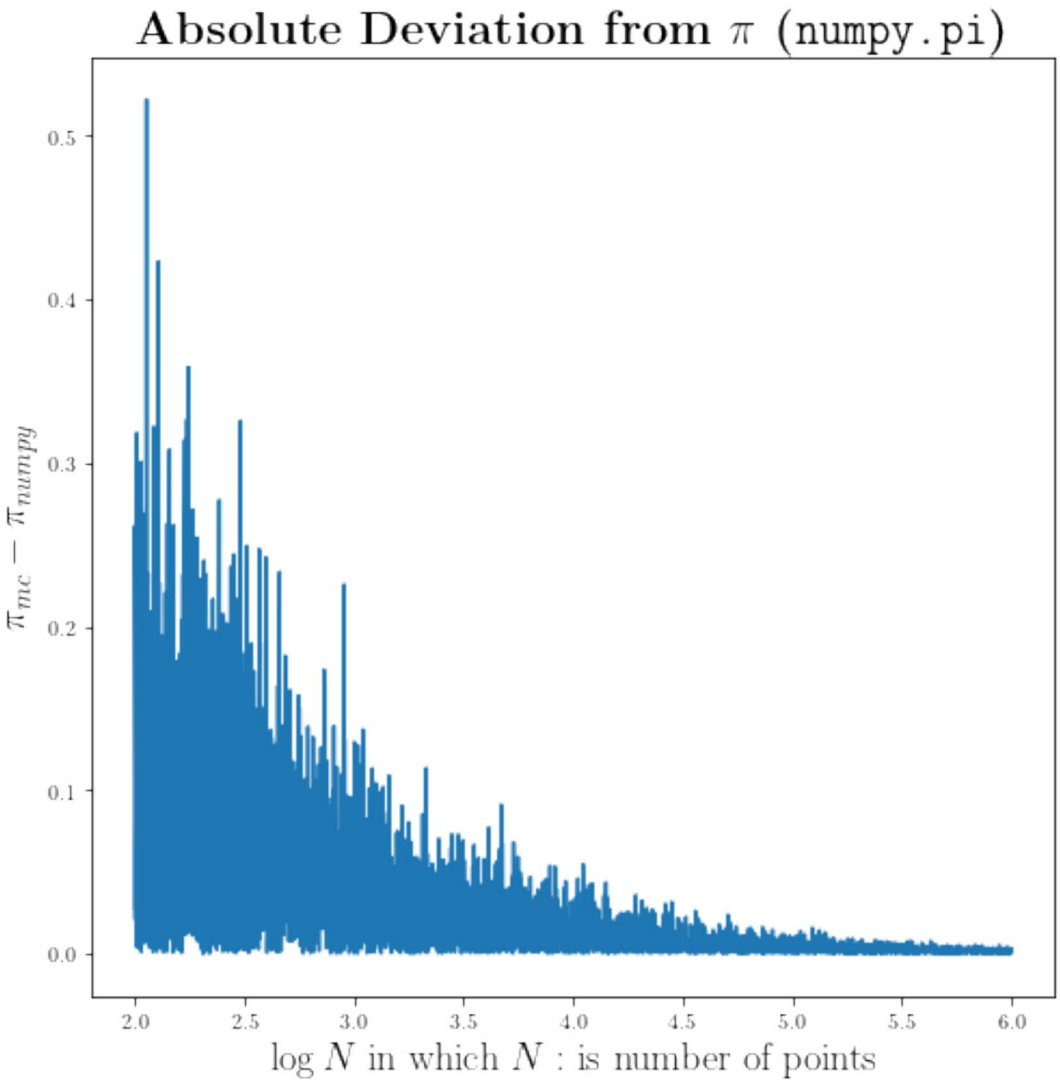
```
In [21]: distance_from_pi = np.array(_x_)-np.pi
```

This is how our estimate errors change with different number of points.

```
In [22]: fig2 = plt.figure(figsize=(8,8))
         plt.plot(np.log10(I), distance_from_pi)

         plt.title(r"\textbf{Deviation from $\pi$ (\texttt{numpy.pi})}", fontsize=22)
         plt.xlabel(r"$\log{N}$ in which $N: $ is number of points", fontsize=18)
         plt.ylabel(r"$\pi_{mc}-\pi_{numpy}$", fontsize=18)

         plt.show()
```




```
In [23]: fig2 = plt.figure(figsize=(8,8))
         plt.plot(np.log10(I), abs(distance_from_pi))

         plt.title(r"\textbf{Absolute Deviation from $\pi$ (\texttt{numpy.pi})}", fontsize=22)
         plt.xlabel(r"$\log{N}$ in which $N: $ is number of points", fontsize=18)
         plt.ylabel(r"$\pi_{mc}-\pi_{numpy}$", fontsize=18)

         plt.show()
```



4.1.3 And as you would expect we are getting better and better by adding more points

Let's put the selection criteria of the circle inside a function. (This can be generalize easily to any arbitrary geometric criteria)

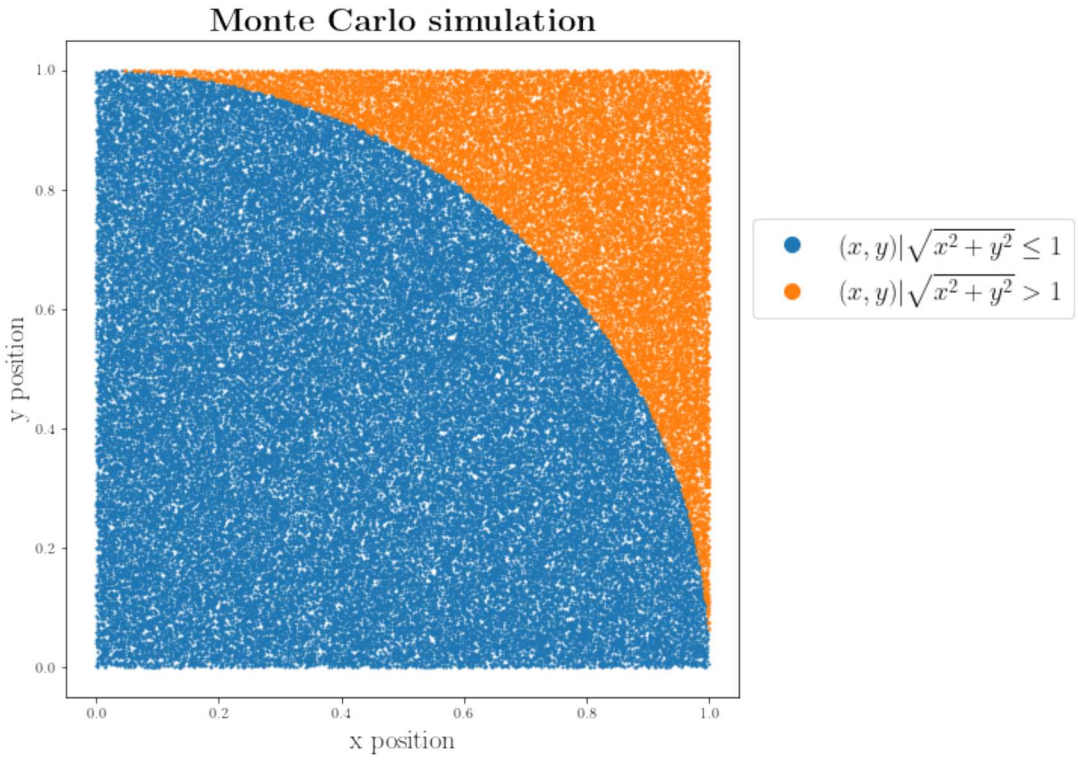
```
In [24]: def impose_circle(distances, radius=1):
             """
             This is the function that takes an array `[distances]`
             and a number (radius) and output an array of the similar
             size, and for those value<radius, assigns 1 and the rest 0
             """
             try:
                 lenght_of_array = len(distances)
```

```
        except TypeError:
            lenght_of_array = 1

        selector=np.zeros(lenght_of_array)
        for i in range(lenght_of_array):
            if distances[i]<=radius:
                selector[i] = 1
        return selector
```

Now let's divide our points into two dictionaries: `inside_points` and `outside_points`

```
In [25]: selector = impose_circle(distance_from_0_0)

         inside_points, outside_points = {}, {}

         inside_points["x"] = x[selector==1]
         inside_points["y"] = y[selector==1]

         outside_points["x"] = x[selector==0]
         outside_points["y"] = y[selector==0]
```

```
In [26]: fig = plt.figure(figsize=(8,8))

         plt.plot(inside_points["x"], inside_points["y"], '.', markersize=2,
                  label=r"$(x,y) | \sqrt{x^2+y^2} \leq 1 $")
         plt.plot(outside_points["x"], outside_points["y"], '.', markersize=2, label=r"$(x,y)

         plt.title(r"\textbf{Monte Carlo simulation}", fontsize=22)

         plt.xlabel(r"x position", fontsize=18)
         plt.ylabel(r"y position", fontsize=18)

         plt.legend(bbox_to_anchor=(1, 0.75), fontsize=18, markerscale=10)


         plt.show()
```



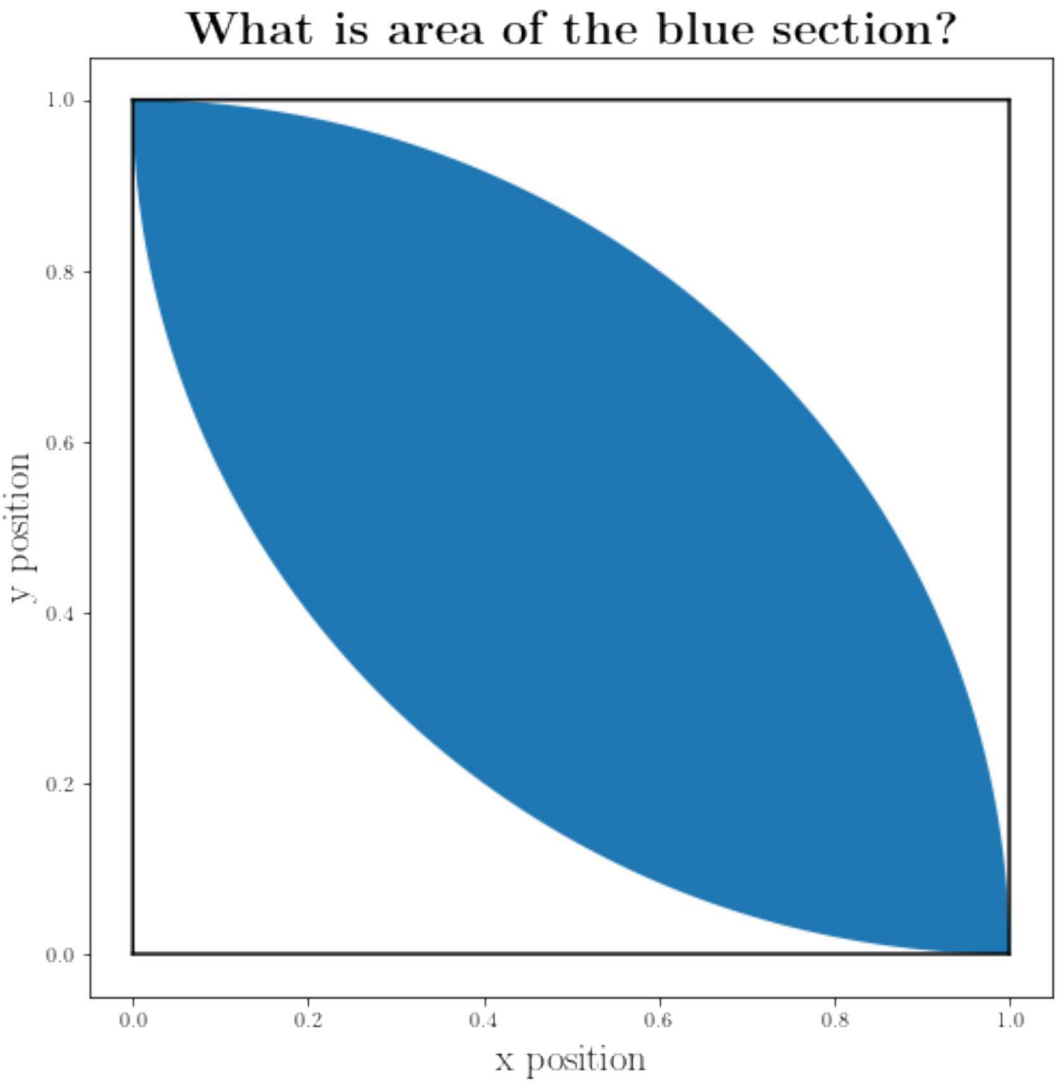
4.1.4 This is how our Monte Carlo simulation actually look like!

4.1.5 Finding the area using Monte Carlo simulation:

In [27]:
```
fig_pi = plt.figure(figsize=(8,8))
```

```
from matplotlib.patches import Rectangle
_number_of_points_=10**6

x = np.linspace(0,1, _number_of_points_)
# defing our curves (circle R^2 = X^2 + Y^2)
y = np.sqrt(1-x**2)
_y_ = 1 - np.sqrt(1-(x-1)**2)
# Fill the area between two circles
plt.fill_between(x,y,_y_)

# Making the square
y1 = np.ones(10**6)
y2 = np.zeros(10**6)

# make the square plot
plt.plot(x, y1, "k")
```

```
plt.plot(x, y2, "k")
plt.plot(y2, x, "k")
plt.plot(y1, x, "k")


plt.title(r"\textbf{What is area of the blue section?}", fontsize=22)

plt.xlabel(r"x position", fontsize=18)
plt.ylabel(r"y position", fontsize=18)


plt.show()
```

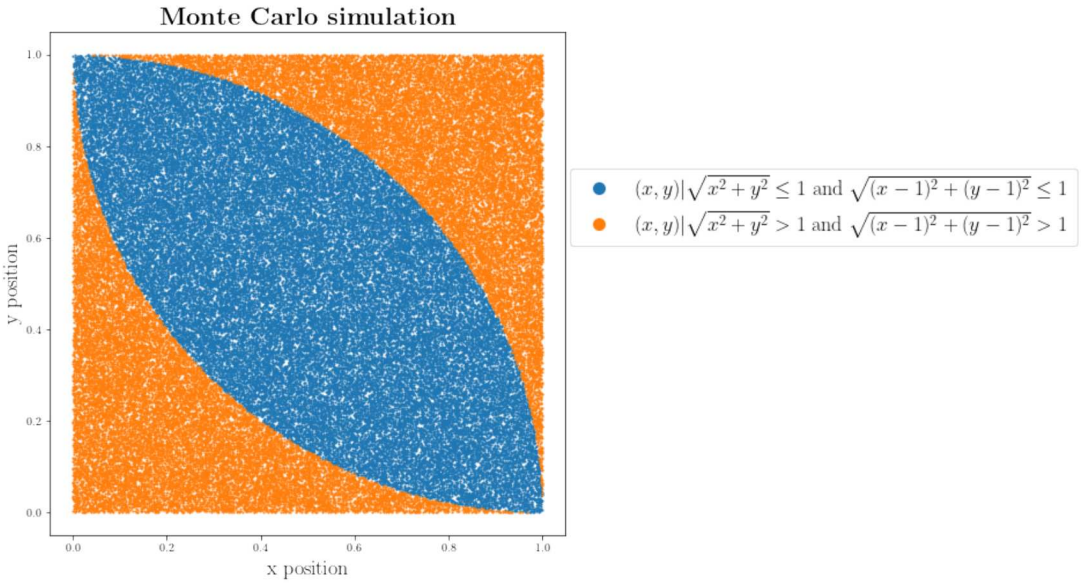


Let's do some **Monte Carlo** simulation again:

```
In [28]: N = 10e4
         N = int(N)

         # Points from (0,0)
         x = np.random.random(N)
         y = np.random.random(N)

         dis=np.sqrt(x**2+y**2)

         # Points frome (1,1)
         x_1= np.ones(N)-x
         y_1= np.ones(N)-y

         dis1=np.sqrt(x_1**2+y_1**2)
```

Now that we have our points let's apply the criteria:

```
In [29]: selector=impose_circle(dis)
         selector1=impose_circle(dis1)

In [30]: final_sel = selector1*selector

In [31]: _inside_points, _outside_points = {}, {}

         _inside_points["x"] = x[final_sel==1]
         _inside_points["y"] = y[final_sel==1]

         _outside_points["x"] = x[final_sel==0]
         _outside_points["y"] = y[final_sel==0]

In [32]: fig = plt.figure(figsize=(8,8))

         plt.plot(_inside_points["x"], _inside_points["y"], '.', markersize=2, label=r"$(x,y)
         plt.plot(_outside_points["x"], _outside_points["y"], '.', markersize=2, label=r"$(x,y

         plt.title(r"\textbf{Monte Carlo simulation}", fontsize=22)

         plt.xlabel(r"x position", fontsize=18)
         plt.ylabel(r"y position", fontsize=18)

         plt.legend(loc='upper center', bbox_to_anchor=(1.5, 0.75), fontsize=18, markerscale=1

         plt.show()
```




```
In [33]: area_of_middle = len(_inside_points["x"])/len(x)
         area_of_middle
```

```
Out[33]: 0.5708
```

4.1.6 Finding the Area geometrically:

Let's find the area for the part of square that we took fourth of a circle (A_{extra}) from and take two of them from the area of a square.

$A_{extra} = A_{square} - \frac{A_{circle}}{4}$

The area that we are looking for becomes:

$A_{shape} = A_{square} - 2A_{extra}$

Let's plug in known quantities: $A_{shape} = R^2 - 2(R^2 - \frac{\pi R^2}{4})$

$$A_{shape} = R^2(\frac{\pi}{2} - 1)$$

for special case of $R = 1$ it becomes:

$$A_{shape} = \frac{\pi}{2} - 1$$

Coding directly to find the area: (above calculation is inside the code)

```
In [34]: area_curve = 1 - np.pi/4
         two_curved_area = 2*area_curve

         area_of_middle_geometry = 1 - two_curved_area
         area_of_middle_geometry
```

```
Out[34]: 0.5707963267948966
```

Using the formula directly:

```
In [35]: np.pi/2-1
```

```
Out[35]: 0.5707963267948966
```

4.1.7 And as you would expect all of these methods are giving us consistent results.

5 Random Walk: As a simple example of modeling a random process

5.1 We are going to make a random walk and try to answer basic questions like what is the expected distance from the staring point, path, ...

There are few ways of appraoching this problem: 1. **Functional approach**: Building the whole process as a pipeline of differnet functions, which is the approach we used so far. 2. **Object oriented approach**: Which is concepetually a very differnet approach but by doing this excersise we'll learn why this approach can be very useful for some problems.

Tip: Desiding what appraoch to take for a particular problem, depends on many factors. One of the easiest factor which is mainly independent of the problem is that whether you are going to reuse your codes again or adding differnet features to it later. If that's the case, generally speaking it is better to try to think about the problem and implement your code with the object oriented approach.

5.1.1 Like always let's start with the simplest case: 1-d random walk

```
In [36]: def random_walk_1d(n, step=1):
             """This is a function for making a 1-d random walk
             INPUT:
                   n (int): number of steps to take
                   step (float): lenght of each steps

             OUTPUT:
                   positions (numpy.array): an array of different positions during
                 the random walk

             """
             import random
             import numpy as np
             # making an array for putting all the information
             positions=np.zeros(n)

             # initial position
             x = positions[0]

             for i in range(1,n):
                 # choosing the random step to take
                 dx = random.choice([1,-1])*step
```